Determination of Unsteady Supersonic Flows around Thin Pointed Wings by Asymptotic Expansions

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Harmonic oscillations of a thin wing in a supersonic flow are considered. For an unsteady flow, we generalize a method that has proved successful in the steady case. The asymptotic expansions so obtained are expressed in terms of a parameter ϵ, which represents the width of the wing divided by its length (the series converge faster as the wing is more slender). A systematic approach permits the determination of the flowfield around a pointed wing, with subsonic leading edge, up to any order in ϵ.

I. Introduction

The steady supersonic flow around a thin wing has been studied in the case of a slender body[1,2] and then generalized for the not-so-slender body.[3] More recently, a systematic approach[4,5] using asymptotic expansions has been developed for wings of arbitrary shapes. The expansions are given in terms of a parameter that represents the width of the wing divided by its length. The method[4,5] permits the determination of the steady flowfield to any order. The first terms of the expansion correspond to the slender and not-so-slender body theories.[1,2,3]

A corresponding slender body theory is also known for unsteady flows around thin wings.[6,7] Following the lines of the general steady theory,[4,5] it is the purpose of this paper to determine the unsteady flowfield to any order.

In principle, one could reduce an unsteady problem to a steady one using the Magnaradze-Galin transformation.[8] However, a solvable steady problem corresponds to a given vibrating wing only if the upper and lower surface of the vibrating wing remain symmetric to each other during their entire motion (symmetric problem). Also, applying the transformation to a known steady flow, one can deduce the solution of a corresponding unsteady problem. Unfortunately, unsteady flows over wings with two different shapes correspond to the same steady flow, to which the transformation is applied at two different frequencies. Hence, the transformation cannot be used in a general and systematic way and is not convenient to apply. Furthermore, the direct method we are going to develop for the study of unsteady flows is as easy to use as in the steady case, so there is no advantage in reducing our problem to a steady one.

We consider a uniform flow slightly perturbed by the oscillations of a wing, i.e., the wing is almost plane. We take as origin the apex of the wing (see Fig. 1), and the wing remains close to the plane $x_3 = 0$. We consider only harmonic motions for the wing (by combination, any periodic motion can be studied). The flow can be decomposed in two parts. The first represents the steady flow around the wing in its average position, the second is the purely harmonic flow. We already know how to compute the steady part of the flow[4,5] and shall not consider it here. For the unsteady part of the flow, the position in time of the upper wing is

$$x_3/b = eg(x_1/l, x_2/b) \exp i\nu\vartheta \tag{1}$$

l is the length and b the width of the wing (see Fig. 1); ν is the frequency, ϑ the time, and e is a small parameter (the problem is linear) of order the maximum thickness of the wing divided by b. The average position of the upper wing lies in the plane $x_3 = 0$ (since we do not consider the steady part of the flow). As usual in wing theory, we shall decompose our problem further in a symmetric and an antisymmetric flow,[6] which by linear combination give the most general flow. In the symmetric flow, the lower surface remains symmetric of the upper surface with respect to $x_3 = 0$ at all times (oscillating thickness case). In the antisymmetric case, the upper and lower surfaces remain identical (lifting case). The wing is taken as symmetric with respect to $x_2 = 0$. The average positions (in the plane $x_3 = 0$) of the leading and trailing edges are defined, respectively, by the equations,

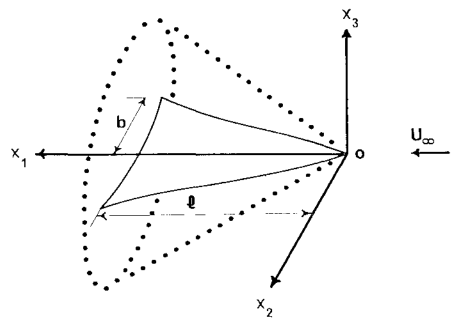

Fig. 1 Sketch of wing and Mach cone at the origin.

$$|x_2/b| = h(x_1/l) \qquad |x_2/b| = k(x_1/l) \tag{2}$$

(for the symmetric problem, the leading and trailing edges remain in the plane $x_3 = 0$ at all times).

For simplicity, we take the time dependence of the harmonic motion as $\exp i\nu\vartheta$. Of course, only the real part of the solution will represent a physical flow (corresponding to wing shapes varying as $\cos\nu\vartheta$). The potential of the flow Φ is written

$$\Phi = lU_\infty\varphi \exp i\nu\vartheta \tag{3}$$

where U_∞ is the unperturbed velocity at infinity upstream. φ satisfies the equation[6]

$$\frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2} = \beta^2\frac{\partial^2\varphi}{\partial x_1^2} + \frac{2iM^2\nu}{U_\infty}\frac{\partial\varphi}{\partial x_1} - \frac{\nu^2M^2}{U_\infty^2}\varphi \tag{4}$$

M is the Mach number of the flowfield, and $\beta^2 = M^2 - 1$. (β is of order one; the flow is supersonic, neither transonic nor hypersonic.) Consider the Mach cone with apex at the origin (Fig. 1) satisfying the equation,

$$x_1 = \beta(x_2^2 + x_3^2)^{1/2} \tag{5}$$

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We assume that the wing is entirely contained within the cone, i.e., the leading edge is subsonic. Finally, we assume the trailing edge to be supersonic so that the wake does not affect the flow over the wing. We shall ignore entirely the flow in the wake and its domain of dependence.

Besides Eq. (4), the potential must satisfy certain boundary conditions. In the region upstream of the Mach cone [given by Eq. (5)], the flow is undisturbed; hence, $\varphi = 0$ there. It is well known[1,6] that φ and its first-order derivatives are continuous across the Mach cone (in general, the second derivatives are discontinuous). On the wing, the velocity must satisfy the slip condition, which, as usual, is written for $x_3 = 0$, since the problem is linear:

$$l(\partial\varphi/\partial x_3) = eb[(\partial g/\partial x_1) + (i\nu/U_\infty)g] \quad (6)$$

In the following it will be convenient to use dimensionless space variables t, x, y defined by $x_1 = lt$, $x_2 = bx$, $x_3 = by$, and let $\lambda = b/l$; $\epsilon = \beta\lambda$. We introduce a reduced potential $f(t,x,y)$ by

$$\varphi = (1/2\pi)e\lambda^2 f \exp - iM\omega t \quad (7)$$

where ω is the reduced frequency $\nu lM/\beta^2 U_\infty$. One interest in using f instead of φ is that f satisfies a real equation, instead of Eq. (4):

$$(\partial^2 f/\partial x^2) + (\partial^2 f/\partial y^2) = \epsilon^2[(\partial^2 f/\partial t^2) + \omega^2 f] \quad (8)$$

with the boundary conditions

$$f = \partial f/\partial t = \partial f/\partial x = \partial f/\partial y = 0 \quad (9)$$

$$\text{for } t \leq \epsilon(x^2 + y^2)^{1/2}$$

$$\partial f/\partial y = 2\pi(\partial G/\partial t) \qquad \text{on the wing} \quad (10)$$

where

$$\partial G/\partial t \equiv [(\partial g/\partial t) + i(\beta^2\omega g/M)] \exp iM\omega t \quad (11)$$

$\partial G/\partial t$ is known when g is given. Note that for $\omega = 0$, our problem reduces to the steady case.[4,5] The geometry and the conditions we imposed on the wing correspond to those of a supersonic transport, which, of course, is currently of considerable interest.

II. Asymptotic Expansion—Generalities

We shall write our solution as an expansion in ϵ. From its definition and the fact that the leading edge is subsonic, we have $\epsilon < 1$. The number of terms we include will depend, in practice, on the magnitude of ϵ. As the wing becomes more slender, ϵ smaller, fewer terms are necessary. In particular, the slender body theory represents the limiting case for a sufficiently small ϵ when only the first one or two terms are significant. When ϵ is not so small (case of a supersonic transport for instance), more terms have to be kept to provide an adequate solution. When $\epsilon \to 0$, l being fixed and letting $b \to 0$, then x and y remain finite near the wing but become infinite near the Mach cone. x and y are the inner variables[9] of our problem. To find the solution near the Mach cone, outer variables[9] will be used, defined by $X = \epsilon x$ and $Y = \epsilon y$. X and Y remain finite near the Mach cone when $\epsilon \to 0$. We shall also use cylindrical coordinates; inner variables, $t, r, \theta(x = r\cos\theta, y = r\sin\theta)$; and outer variables, $t, R, \theta(R = \epsilon r)$. The procedure used with asymptotic expansions[9] consists of writing the most general outer and inner solutions. The former satisfies the boundary conditions on the Mach cone; the latter, on the wing. Remaining unknown coefficients will be determined by matching the inner and outer solutions. The matching principle will be used in the form[9]:

inner representation of "outer solution" =

outer representation of "inner solution"

In order to write the most general expansions for the inner and outer solutions, it is of fundamental importance to know the set of gage functions[9] which will constitute the asymptotic sequence.

III. Outer Solution and Gage Functions

Equation (8) rewritten in terms of outer variables given at once

$$(\partial^2 f/\partial X^2) + (\partial^2 f/\partial Y^2) = (\partial^2 f/\partial t^2) + \omega^2 f \quad (12)$$

The parameter ϵ does not appear explicitly in the outer equation. Let $\langle f\rangle$ be the Laplace transform of f, $\langle f\rangle = \mathcal{L}(f)$ or

$$\langle f\rangle = \int_0^\infty e^{-pt} f(t)dt \quad (13)$$

$\langle f\rangle$ satisfies the outer equation [from Eq. (9) and Eq. (12)],

$$\Delta\langle f\rangle = (\omega^2 + p^2)\langle f\rangle \quad (14)$$

In cylindrical coordinates and by separation of variables, Eq. (14) has for elementary solution $\langle f_n\rangle$

$$\langle f_n\rangle = \mathrm{K}_n[(p^2 + \omega^2)^{1/2}R]\{A_n \cos n\theta + B_n \sin n\theta\} \quad (15)$$

n is a positive integer, and K_n is the modified Bessel function of the second kind. The general outer solution is a linear combination of elementary solutions (for every n) which is known when all $A_n(p)$ and $B_n(p)$ are known. Note that I_n, the modified Bessel function of the first kind, does not enter in the solution as it would lead to perturbations becoming infinite far from the wing. Through the matching principle, A_n and B_n will be found and will depend on ϵ. If we assume for an instant that we know this dependence, the inner representation of the outer solution is obtained by replacing R by ϵr in Eq. (15) and rearranging the linear combination of elementary solutions in an asymptotic sequence with gage functions of increasing order. To do so, we first need the expansion of K_n introduced into Eq. (15),

$$\mathrm{K}_n[\epsilon r(p^2 + \omega^2)^{1/2}] = \sum_{l=0}^{\infty} \mathrm{K}_{n,l}[(p^2 + \omega^2)^{1/2}, r] \times \left(\frac{\epsilon}{2}\right)^{2l-n} - \sum_{l=0}^{\infty} \tilde{\mathrm{K}}_{n,l}[(p^2 + \omega^2)^{1/2}, r] \left(\frac{\epsilon}{2}\right)^{2l-n} \ln\frac{\epsilon}{2} \quad (16)$$

$$\mathrm{K}_{n,l} = \frac{1}{2}(-1)^l \frac{(n-l-1)!}{l!} r^{2l-n}(p^2 + \omega^2)^{(2l-n)/2} \quad (17)$$

$$n > l$$

$$\mathrm{K}_{n,l} = \frac{(-1)^n}{l!(l-n)!}\left[\frac{1}{2}(s_l + s_{l-n}) - \ln(p^2 + \omega^2)^{1/2} r - \gamma\right] \times r^{2l-n}(p^2 + \omega^2)^{(2l-n)/2} \qquad l \geq n \quad (17a)$$

$$\tilde{\mathrm{K}}_{n,l} = 0 \qquad n > l \quad (18)$$

$$\tilde{\mathrm{K}}_{n,l} = (-1)^n \frac{r^{2l-n}}{l!(l-n)!}(p^2 + \omega^2)^{(2l-n)/2} \qquad l \geq n \quad (18a)$$

where γ is Euler's constant and

$$s_l = \sum_{q=1}^{l} \frac{1}{q} \qquad s_0 = 0$$

The inner solution introduces a gage function of order ϵ^0 through the boundary condition on the wing. The differential equation (8) will relate two elementary solutions corresponding to gage functions that differ in order by ϵ^2. Hence, by iteration, any gage function will introduce an infinite number of gage functions differing from the first by orders $\epsilon^2, \epsilon^4, \ldots, \epsilon^{2n}, \ldots$. We are now going to find the complete set of gage functions to be used in the symmetric and antisymmetric problems.

Symmetric Problem: $A_n \neq 0, B_n = 0$

Using the expansion for K_0, [Eq. (16)], the outer representation of the inner solution contains terms of the form

A_0, $A_0 \ln\epsilon$. A_0 in general will contain a term in ϵ^0 necessary to represent the behavior of the function of order ϵ^0 which satisfies the boundary condition on the wing for large r. By iteration, using the differential equation, the gage functions will include terms of the form

$$(\epsilon/2)^{2m} \ln^{\mu}\epsilon/2 \qquad m \geq 0 \qquad \mu = 0,1 \tag{19}$$

Actually, for the symmetric problem, all gage functions are included in Eq. (19). The fact that no other gage functions are necessary to find the complete solution of our problem will be apparent in the next section. By using the gage functions from Eq. (19), we shall be able to find an expression satisfying the differential equation and the boundary conditions exactly. Since the solution to the problem is unique, no additional gage function needs to be considered. Heuristically, we can easily show why it is so without going into the details of the next section. The term $A_1K_1 \cos\theta$ by expansion contains terms of the form $A_1 \cos\theta r^{-1}\epsilon^{-1}$ and $A_1 \cos\theta r\epsilon \ln\epsilon$. A_1 must start by a term of order ϵ so that the first term might contribute to the boundary condition on the wing. The second term is then of the form $\cos\theta r\epsilon^2 \ln\epsilon = x\epsilon^2 \ln\epsilon$ which has the form prescribed by Eq. (19). Furthermore, the latter term does not perturb the boundary condition on the wing. From Eq. (19), we deduce at once that A_n can be written as

$$A_n(\epsilon,p) = \left(\frac{\epsilon}{2}\right)^n \sum_{m=0}^{\infty} \left(\frac{\epsilon}{2}\right)^{2m} \sum_{\mu=0,1} \ln^{\mu} \frac{\epsilon}{2} A_{nm\mu}(p) \tag{20}$$

Antisymmetric Problem: $A_n = 0$, $B_n \neq 0$

Consider the expression $B_1K_1 \sin\theta$, which by expansion gives terms of the form $B_1 \sin\theta r^{-1}\epsilon^{-1}$ and $B_1 \sin\theta r\epsilon \ln\epsilon$. Once more, the first term is needed to satisfy the wing condition; hence, B_1 starts with a term in ϵ. The second term introduces the function $\sin\theta r\epsilon^2 \ln\epsilon = y\, \epsilon^2\ln\epsilon$ which perturbs the boundary conditions on the wing. The perturbation must be cancelled, which is done easily by repeating the process with a boundary condition of order $\epsilon^2 \ln\epsilon$. In that case, B_1 must also contain a term of order $\epsilon^3\ln\epsilon$, and $B_1 \sin\theta r\epsilon \ln\epsilon$ introduces a gage function of order $\epsilon^4 \ln^2\epsilon$. By repetition of the process, it is quite obvious that for the antisymmetric problem the most general gage function is

$$(\epsilon/2)^{2m} \ln^{\mu}\epsilon/2 \qquad 0 \leq \mu \leq m \tag{21}$$

Consequently

$$B_n(\epsilon,p) = \left(\frac{\epsilon}{2}\right)^n \sum_{m=0}^{\infty} \left(\frac{\epsilon}{2}\right)^{2m} \sum_{\mu=0}^{m} \ln^{\mu}\frac{\epsilon}{2} B_{nm\mu}(p) \tag{22}$$

$$n \geq 1$$

IV. Symmetric Problem

From the foregoing section, the Laplace transform of f can be written as

$$\langle f\rangle = \sum_{\nu=0}^{\infty} \left(\frac{\epsilon}{2}\right)^{2\nu} \sum_{\mu=0,1} \langle f_{\nu\mu}\rangle \ln^{\mu} \frac{\epsilon}{2} \tag{23}$$

where, from Eqs. (16, 17 and 18)

$$\langle f_{\nu0}\rangle = \left\{\sum_{l=0}^{\nu} \sum_{n=0}^{l} \frac{(-1)^n}{l!(l-n)!} \left[\frac{1}{2} (s_l + s_{l-n}) - \ln(p^2 + \omega^2)^{1/2} - \gamma\right] r^{2l-n}(p^2 + \omega^2)^{(2l-n)/2} + \sum_{l=0}^{\nu} \sum_{n=l+1}^{\infty} \frac{1}{2} (-1)^l \frac{(n-l-1)!}{l!} r^{2l-n}(p^2 + \omega^2)^{(2l-n)/2}\right\} \times A_{n,\nu-l,0} \cos n\theta \tag{24}$$

$$\langle f_{\nu1}\rangle = -\sum_{l=0}^{\nu} \sum_{n=0}^{l} \frac{(-1)^n}{l!(l-n)!} r^{2l-n}(p^2 + \omega^2)^{(2l-n)/2} \times A_{n,\nu-l,0} \cos n\theta \tag{24a}$$

Or by inversion of Eqs. (23) and (24),

$$f = \sum_{\nu=0}^{\infty} (\epsilon/2)^{2\nu} \sum_{\mu=0,1} f_{\nu\mu} \ln^{\mu} \frac{\epsilon}{2} \tag{25}$$

where

$$f_{\nu0} = \left\{\sum_{l=0}^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n r^{2l-n}}{l!(l-n)!} \left\{\left[\frac{1}{2} (s_l + s_{l-n}) - \ln r\right] \times \alpha^{\{2l\}}{}_{n,\nu-l,0} + a^{\{2l\}}{}_{n,\nu-l,0}\right\} + \sum_{l=0}^{\nu} \sum_{n=l+1}^{\infty} \frac{1}{2} (-1)^l \times \frac{(n-l-1)!}{l!} r^{2l-n} \alpha^{\{2l\}}{}_{n,\nu-l,0}\right\} \cos n\theta \tag{26}$$

$$f_{\nu1} = -\sum_{l=0}^{\nu} \sum_{n=0}^{l} (-1)^n \frac{r^{2l-n}}{l!(l-n)!} \alpha^{\{2l\}}{}_{n,\nu-l,0} \cos n\theta \tag{26a}$$

By definition

$$\alpha_{n\nu0} = \mathcal{L}^{-1}[(p^2 + \omega^2)^{-n/2} A_{n\nu0}]$$

$$a_{n,\nu-l,0} = \int\!\!\!\!\!\!-_{0}^{t} \frac{\cos\omega\vartheta}{\vartheta} \alpha_{n,\nu-l,0}(t - \vartheta)d\vartheta$$

$\displaystyle\int\!\!\!\!\!-$ represents Hadamard's finite part of the integral.[10] In Eq. (26) and in the following, we use the notation

$$f^{\{2l\}} \equiv \left(\frac{\partial^2 f}{\partial t^2} + \omega^2\right)^{(l)}$$

Equation (25) is the inner representation of the most general outer solution. From the matching principle, it also represents the outer representation of the inner solution, which we are now going to determine. Equation (8) is the differential equation satisfied by the inner solution. One can check directly, by a tedious but straightforward computation, that f, as given by Eqs. (25) and (26), satisfies Eq. (8) to all orders, i.e.,

$$\Delta f_{00} = \Delta f_{01} = 0 \qquad \Delta f_{\nu\mu} = 4\left(\frac{\partial^2}{\partial t^2} + \omega^2\right) f_{\nu-1,\mu} \tag{27}$$

$$\mu = 0,1 \qquad \nu \geqslant 1$$

f, which is the outer representation of (inner solution), written with inner variables, satisfies also the inner equation [Eqs. (8) or (27)]. Hence f, as given by Eqs. (25) and (26), is the inner solution written in a form appropriate for large r. We can say that the outer representation of the inner solution is simply the expansion of the inner solution for large r. That result, which may seem quite intuitive, is not true in general when asymptoptic methods are used.[9] The fundamental reason why it holds in the present problem is that the inner solution satisfies Laplace and Poisson equations [Eq. (27)]. As is well known (see also below), the solution of such equations can be expanded for large r using only powers and logarithms of r, which in the outer representation will only introduce powers and logarithms of ϵ. Hence, writing the outer representation of inner solution terms are simply rearranged (none is lost). The inner solution is of the form given by Eq. (25), but the $f_{\nu\mu}$ that appears in it will be written in a form appropriate to satisfy the wing condition. It will reduce to the form given by Eq. (26) only by expanding it for large r. No confusion should arise, as we keep, for simplicity, the same $f_{\nu\mu}$ to represent both expressions. We must find $f_{\nu\mu}$ near the wing, satisfying the boundary condition on it, and also Eq. (27). It will be convenient to use for new variables

$$z = x + jy \qquad \bar{z} = x - jy$$

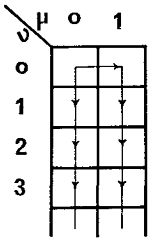

Fig. 2 Schematic representation of the procedure for a symmetric wing.

We consider functions $\mathcal{L}_{\nu\mu}(t,z,\bar{z})$ the real part of which† is

$$f_{\nu\mu} = \frac{Re}{J}\{\mathcal{L}_{\nu\mu}(t,z,\bar{z})\}$$

Eq. (27) reduces to

$$\mathcal{L}_{0\mu} = \mathcal{L}_{0\mu}(t,z) \qquad \mu = 0,1$$

$$\frac{\partial^2 \mathcal{L}_{\nu\mu}(t,z,\bar{z})}{\partial z \partial \bar{z}} = \left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\mathcal{L}_{\nu-1,\mu} \tag{28}$$

$$\nu \geqslant 1 \qquad \mu = 0,1$$

Once $\mathcal{L}_{00}$ and $\mathcal{L}_{01}$ are known ($\mathcal{L}_{00}$ will satisfy the wing condition), all other functions will be determined by iteration up to a harmonic function. Those harmonic functions (as well as $\mathcal{L}_{01}$) must satisfy the following conditions: 1) the boundary condition on the wing is not perturbed, and 2) $f_{\nu\mu}$ can be identified for large r with the expression given in Eq. (26). Let us write

$$\mathcal{L}_{00} = \mathcal{P}_{00} + \mathcal{Q}_{00}$$

with

$$\mathcal{P}_{00} = 2\int_{-h}^{h} \ln(z-\xi)\frac{\partial G}{\partial t}(t,\xi)d\xi$$

and

$$\mathbf{P}_{00} = \frac{Re}{J}(\mathcal{P}_{00})$$

$\mathbf{P}_{00}$ obviously satisfies the boundary condition on the wing. $\mathcal{Q}_{00}$ again is a harmonic function that does not perturb the wing condition and is necessary for identification (for large r) with the expression in Eq. (26). Similarly let us write $\mathcal{L}_{\nu0} = \mathcal{P}_{\nu0} + \mathcal{Q}_{\nu0}$ with

$$\mathcal{P}_{\nu0} = 2\left[\frac{\partial^2}{\partial t^2} + \omega^2\right]^{(\nu)}\int_{-h}^{h} L_\nu(z-\xi)\frac{\partial G}{\partial t}\frac{(\bar{z}-\xi)^\nu}{\nu!}d\xi \tag{29}$$

$$\nu \geqslant 1$$

where $L_\nu(x) = x^\nu/\nu!\,(\ln x - s_\nu)$. $\mathcal{P}_{\nu0}$ is so defined that it satisfies Eq. (28); also it does not perturb the wing condition since

$$\partial\mathbf{P}_{\nu0}/\partial y \equiv \frac{Re}{J}J[(\partial\mathcal{P}_{\nu0}/\partial z) - (\partial\mathcal{P}_{\nu0}/\partial\bar{z})]$$

is zero for $y = 0$, $\nu > 0$. $\mathcal{Q}_{\nu0}$ plays the same role as $\mathcal{Q}_{00}$. Let us now expand

$$P_{\nu0} = \frac{Re}{J}(\mathcal{P}_{\nu0}) \qquad \nu \geqslant 0$$

and compare it with the expansion of $f_{\nu0}$ from Eq. (26). By difference, we shall deduce the expansion of

$$Q_{\nu0} = \frac{\mathrm{Re}}{J}(\mathcal{Q}_{\nu0})$$

† Remark that i and j are different symbols, $i^2 = j^2 = -1$ but $ij \neq -1$. The real part of $\mathcal{L}_{\nu\mu}$ may contain i [the operation represented by $R_e(\mathcal{L}_{\nu\mu})$ is only carried on j].

We find at once by expansion of $P_{\nu0}$ for large r that

$$\alpha_{n\nu0} = \frac{-2\epsilon_0^n}{\nu!(\nu+n)!}\left[\frac{\partial^2}{\partial t^2} + \omega^2\right]^{(\nu)}\int_{-h}^{h}\xi^{2\nu+n}\frac{\partial G}{\partial t}d\xi \tag{30}$$

where $\epsilon_0^0 = 1$; $\epsilon_0^n = 2$, $n \neq 0$. Since the wing we consider is symmetric with respect to $x_2 = 0$,

$$\partial G(t,-\xi)/\partial t = \partial G(t,\xi)/\partial t$$

Hence, Eq. (30) shows that $\alpha_{2p+1,\nu,0} = 0$. We also find that for large r

$$Q_{\nu0} = \sum_{l=0}^{\nu}\sum_{n=0}(-1)^n\frac{r^{2l-n}}{l!(l-n)!}a^{\{2l\}}{}_{n,\nu-l,0}\cos n\theta \tag{31}$$

or

$$Q_{00} = \mathcal{Q}_{00} = 2\overline{\left|\int_0^t \frac{\cos\omega(t-\vartheta)}{t-\vartheta}\left[\int_{-h}^{h}\frac{\partial G}{\partial\vartheta}(\vartheta,\mu)d\xi\right]d\vartheta\right.} \tag{32}$$

and by iteration

$$Q_{\nu0} = \mathcal{Q}_{\nu0} = 2\left[\frac{\partial^2}{\partial t^2} + \omega^2\right]^{(\nu)}\times$$

$$\overline{\left|\int_0^t \frac{\cos\omega(t-\vartheta)}{t-\vartheta}\left[\int_{-h}^{h}\frac{\partial G}{\partial\vartheta}\frac{(z-\xi)^\nu}{\nu!}\frac{(\bar{z}-\xi)^\nu}{\nu!}d\xi\right]\right.} \tag{33}$$

One checks at once that the wing condition is not perturbed. From Eq. (26), we deduce that $f_{01} = -\alpha_{000}$ and by iteration

$$f_{\nu1} = 2\left[\frac{\partial^2}{\partial t^2} + \omega^2\right]^{(\nu)}\int_{-h}^{h}\frac{\partial G}{\partial t}\frac{(z-\xi)^\nu}{\nu!}\frac{(\bar{z}-\xi)^\nu}{\nu!}d\xi \tag{34}$$

which, expanded for large r gives

$$f_{\nu1} = -\sum_{l=0}^{\nu}\sum_{n=0}^{l}\frac{(-1)^n r^{2l-n}\cos n\theta\,\alpha^{\{2l\}}{}_{n,\nu-l,0}}{l!(l-n)!} \tag{35}$$

which is exactly of the form prescribed by Eq. (26). Contrary to what happened with $f_{\nu0}$, we do not have to add a subsidiary function similar to $\mathcal{Q}_{\nu0}$. For the symmetric problem, we were able to find the complete solution in closed form up to any order. Figure 2 summarizes the procedure employed as indicated by the arrows. Vertical arrows represent the iteration, and the horizontal arrow (deducing f_{10} from f_{00}) comes from the behavior of the inner solution for large r. Note that the determinations of the two columns are independent of each other, except for f_{00} and f_{01}. The first two terms of our solution ($\nu = 0$) correspond to the slender body theory and the first four terms ($\nu = 0,1$) would correspond to the not-so-slender body theory.

V. Antisymmetric Problem

Although the results are quite different, the procedure remains basically that of the previous section and will be presented with fewer details. For the present problem, the inner solution is of the form (see Sec. III),

$$f = \sum_{\nu=0}^{\infty}\left(\frac{\epsilon}{2}\right)^{2\nu}\sum_{\mu=0}^{\nu}f_{\nu\mu}\ln^\mu\left(\frac{\epsilon}{2}\right) \tag{36}$$

$f_{\nu\mu}$ for large r will be of the form [using Eqs. (16, 17, and 18)]

$$f_{\nu\mu} = \left(\sum_{l=0}^{\nu-\mu}\sum_{n=0}^{l}(-1)^n\frac{r^{2l-n}}{l!(l-n)!}\times\right.$$

$$\left\{\left[\frac{1}{2}(s_{l-n} + s_l) - \ln r\right]\beta^{\{2l\}}{}_{n,\nu-l,\mu} + b^{\{2l\}}{}_{n,\nu-l,\mu}\right\} +$$

$$\sum_{l=0}^{\nu-\mu}\sum_{n=l+1}^{\infty}\frac{1}{2}(-1)^l\frac{(n-l-1)!}{l!}r^{2l-n}\beta^{\{2l\}}{}_{n,\nu-l,\mu}$$

$$\left.\sum_{l=0}^{\nu-\mu+1}\sum_{n=0}^{l}(-1)^n\frac{r^{2l-n}}{l!(l-n)!}\beta^{\{2l\}}{}_{n,\nu-l,\mu-1}\right)\sin n\theta \tag{37}$$

Equations (36) and (37) represent the inner form of the outer solution, but again because it is an exact solution of Eq. (8), it is simply the inner solution for large r. In Eq. (37), we have

$$\beta_{n\nu\mu} = \mathcal{L}^{-1}[(p^2 + \omega^2)^{-n} B_{n\nu\mu}]$$

and

$$b_{n,\nu-l,\mu} = \left| \int_0^t \frac{\cos\omega\vartheta}{\vartheta} \beta_{n,\nu-l,\mu}(t-\vartheta)d\vartheta \right.$$

$\mathcal{L}_{\nu\mu}$ (whose real part is $f_{\nu\mu}$) satisfies [from Eq. (8)]

$$\mathcal{L}_{\nu\nu} = \mathcal{L}_{\nu\nu}(t,z)$$

$$\frac{\partial^2 \mathcal{L}_{\nu\mu}}{\partial z \partial \bar{z}} = \left(\frac{\partial^2}{\partial t^2} + \omega^2\right) \mathcal{L}_{\nu-1,\mu} \qquad \nu \geqslant 1,\ \mu \leqslant \nu - 1 \tag{38}$$

Consider the harmonic function

$$\mathcal{P}_{00} = -\int_{-h}^{h} \frac{\partial G}{\partial t}(t,\xi) \ln\left[\frac{(h^2-\xi^2)^{1/2} + (h^2-z^2)^{1/2}}{(h^2-\xi^2)^{1/2} - (h^2-z^2)^{1/2}}\right] d\xi$$

whose real part obviously satisfies the slip condition on the wing. It will be convenient to write $\mathcal{P}_{00}$ under a different but equivalent form. $\mathbf{P}_{00}$ on the wing has the value

$$\tilde{\mathbf{P}}_{00} = -\int_{-h}^{h} \frac{\partial G}{\partial t}(t,\xi) \ln\left|\frac{(h^2-\xi^2)^{1/2} + (h^2-x^2)^{1/2}}{(h^2-\xi^2)^{1/2} - (h^2-x^2)^{1/2}}\right| d\xi \tag{39}$$

We can, then, write

$$\mathcal{P}_{00} = \frac{j}{\pi} \int_{-h}^{h} \frac{\tilde{P}_{00}(t,\xi)}{z-\xi} d\xi \tag{40}$$

From which, we conclude at once by expansion for large r and identification with Eq. (38),

$$\beta_{2p+1,0,0} = \frac{2}{\pi(2p)!} \int_{-h}^{h} \tilde{\mathbf{P}}_{00}\xi^{2p} d\xi \tag{41}$$

because of the symmetry of the wing with respect to ox_1, $\beta_{2p,0,0} = 0$. Also,

$$\mathcal{L}_{00} = \mathcal{P}_{00} \tag{42}$$

No additional terms are needed in the present case to recuperate the expansion for large r.

$\mathcal{L}_{00}$ being known, we are now in a position to determine $\mathcal{L}_{\nu\nu}(t,z)$, $\nu \geqslant 1$. $\mathcal{L}_{\nu\nu}$ must not perturb the boundary condition on the wing and must be of the form prescribed by Eq. (37); in particular, it behaves like $r \sin\theta$ for large r, hence

$$\mathcal{L}_{\nu\nu} = \gamma_{\nu\nu 0}(t)(h^2 - z^2)^{1/2} \tag{43}$$

For large r, this expression gives

$$f_{\nu\nu} = \gamma_{\nu\nu 0}\left[r\sin\theta + \frac{(2p)!h^{2p+2}\sin(2p+1)\theta}{(p!)^2 2^{2p+1}(p+1)r^{2p+1}}\right] \tag{44}$$

which must be identical with the expression deduced from Eq. (37)

$$f_{\nu\nu} = r\sin\theta\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\beta_{1,\nu-1,\nu-1} + \frac{1}{2}\sum_{p=0}^{\infty}(2p)!\frac{\sin(2p+1)\theta}{r^{2p+1}}\beta_{2p+1,\nu,\nu} \tag{45}$$

Hence,

$$\gamma_{\nu\nu 0} = \left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\beta_{1,\nu-1,\nu-1} \tag{46}$$

and

$$\beta_{2p+1,\nu,\nu} = \frac{h^{2p+2}[(\partial^2/\partial t^2) + \omega^2]\beta_{1,\nu-1,\nu-1}}{(p!)^2(p+1)2^{2p}} \qquad \nu \geqslant 1 \tag{47}$$

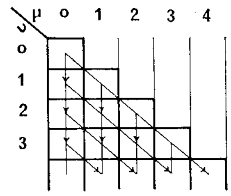

Fig. 3 Schematic representation of the procedure for an antisymmetric wing.

These equations define $\beta_{2p+1,\nu,\nu}$ (and $\mathcal{L}_{\nu\nu}$) since β_{100} is known from Eq. (41). Figure 3 illustrates the method used for the antisymmetric case. The first term $\mathcal{L}_{00}$ is determined (essentially by the slip condition on the wing). From it, we deduced all terms $\mathcal{L}_{\nu\nu}$ on the first diagonal. The next step is to determine the second term in the first column $\mathcal{L}_{10}$, from which we shall deduce the whole second diagonal ($f_{\mu+1,\mu}$). The procedure may then be repeated for the third diagonal and so on. We propose a systematic scheme to determine the solution to any order. Contrary to the symmetric case, one cannot write at once the complete solution in closed form [like Eqs. (29, 33, and 34)]. The solution can be found without difficulty to any order but must be obtained step by step. The computations are straightforward even if somewhat lengthy. The complete calculation of the second diagonal ($\mathcal{L}_{\mu+1,\mu}$) will be given, and we shall indicate the rules to be followed in order to compute the others ($\mathcal{L}_{\mu+\lambda,\mu;\lambda\geqslant 2}$) if necessary. By integration of $\mathcal{P}_{00}$ [Eq. (40)], we deduce a function $\mathcal{P}_{\nu 0}$ satisfying Eq. (38),

$$\mathcal{P}_{\nu 0} = \frac{j}{\pi}\left[\frac{\partial^2}{\partial t^2} + \omega^2\right]^{(\nu)} \int_{-h}^{h} \tilde{\mathbf{P}}_{00}\left[\frac{(\bar{z}-\xi)^{\nu}}{\nu!}L_{\nu-1}(z-\xi) + \frac{(z-\xi)^{\nu-1}}{(\nu-1)!}L_{\nu}(\bar{z}-\xi)\right]d\xi \tag{48}$$

One checks at once that $\partial P\nu 0/\partial y \neq 0$ on the wing; hence, $\mathcal{P}_{\nu 0}$ cannot represent $\mathcal{L}_{\nu 0}$ entirely. Let us add to $P_{\nu 0}$ a term $Q_{\nu 0}$ so that on the wing

$$\partial Q_{\nu 0}/\partial y = -\partial P_{\nu 0}/\partial y$$

which is known from Eq. (48) $\mathcal{Q}_{\nu 0}$ will satisfy Eq. (40) and $\mathcal{Q}_{00} = 0$. Let us determine Q_{10}. We must solve a problem identical to the one solved for P_{00}, or

$$\mathcal{Q}_{10} = \frac{j}{\pi}\int_{-h}^{h}\frac{\tilde{Q}_{10}}{z-\xi}d\xi \tag{49}$$

where

$$\tilde{Q}_{10} = \frac{1}{2\pi}\int_{-h}^{h}\frac{\partial \mathbf{P}_{00}}{\partial y}\ln\left|\frac{(h^2-\xi^2)^{1/2} + (h^2-x^2)^{1/2}}{(h^2-\xi^2)^{1/2} - (h^2-x^2)^{1/2}}\right|d\xi \tag{50}$$

Hence we write

$$\mathcal{L}_{10} = \mathcal{P}_{10} + \mathcal{Q}_{10} + \gamma_{100}(h^2 - z^2)^{1/2} \tag{51}$$

the last term being added for identification with f_{10} as given by Eq. (37) for large r. We find at once by identifying the terms in $r\sin\theta$,

$$\gamma_{100} = -\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)b_{100}$$

or

$$\gamma_{100} = -\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\left|\int_0^t \frac{\cos\omega\vartheta}{\vartheta}\beta_{100}(t-\vartheta)d\vartheta\right. \tag{52}$$

and β_{100} is known from Eq. (41). $\mathcal{L}_{10}$ is now entirely determined. Identifying the terms in $r^{-1}\sin\theta$ for large r we find

$$\beta_{110} = \left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\beta_{300} + \frac{2}{\pi}\int_{-h}^{h}\tilde{Q}_{10}d\xi + h^2\gamma_{100} \tag{53}$$

which will be of use later.

We are now in a position to compute all other functions on the second diagonal ($\mathfrak{L}_{\mu+1,\mu}$; $\mu \geqslant 1$). For large r, we deduce from Eq. (37)

$$f_{\mu+1,\mu} = -r\sin\theta\left[\left(\frac{1}{2}-\ln r\right)\beta^{\{2\}}{}_{1\mu\mu} - \beta^{\{2\}}{}_{1,\mu,\mu-1} + b^{\{2\}}{}_{1\mu\mu}\right] + \frac{r^4\sin\theta}{2}\,\beta^{\{4\}}{}_{1,\mu-1,\mu-1} + \frac{1}{2}\times \sum_{p=0}^{\infty}\frac{(2p)!\,\sin(2p+1)\theta}{r^{2p+1}}\,\beta_{2p+1,\mu+1,\mu} - \frac{r^2}{2}\sum_{p=1}^{\infty}\frac{(2p-1)!\,\sin(2p+1)\theta}{r^{2p+1}}\,\beta^{\{2\}}{}_{2p+1,\mu,\mu} \quad (54)$$

We write the inner solution as

$$\mathfrak{L}_{\mu+1,\mu} = \mathcal{P}_{\mu+1,\mu} + \gamma_{\mu+1,\mu,0}\,(h^2 - z^2)^{1/2} \quad (55)$$

$\mathcal{P}_{\mu+1,\mu}$ is a particular solution of

$$\frac{\partial^2\mathcal{P}_{\mu+1,\mu}}{\partial z\partial\bar{z}} = \left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\mathfrak{L}_{\mu\mu} \quad (56)$$

and $(h^2 - z^2)^{1/2}$ is added to identify the terms in $r\sin\theta$ for large r. Let us take

$$\mathcal{P}_{\mu+1,\mu} = \frac{\gamma_{\mu\mu0}}{2}\,z\bar{z}(h^2 - z^2)^{1/2} + \left(\gamma_{\mu\mu0}\,\frac{h^2}{2}\right)^{\{2\}}\times \bar{z}\left(\sin^{-1}\frac{z}{h} - \sin^{-1}\frac{\bar{z}}{h}\right) + \gamma_{\mu\mu0}\,z(z-\bar{z})\times (h^2 - z^2)^{-1/2}\left[\left(\frac{\partial h}{\partial t}\right)^2 + \omega^2 h\right] \quad (57)$$

We can check at once that such expression is uniform, satisfies Eq. (56), and does not perturb the slip condition on the wing. Identifying with Eq. (54), we deduce

$$(h^2\gamma_{\mu\mu0})^{\{2\}}\,\ln\frac{2}{h} + \gamma^{\{2\}}{}_{\mu\mu0}\,\frac{h^2}{4} + \gamma_{\mu+1,\mu-0} - 2\gamma_{\mu\mu0}\left[\left(\frac{\partial h}{\partial t}\right)^2 + \omega^2h^2\right] = \beta^{\{2\}}{}_{1,\mu,\mu-1} - b^{\{2\}}{}_{1\mu\mu} - \frac{1}{2}\beta^{\{2\}}{}_{1\mu\mu} \quad (58)$$

and

$$\beta_{2p+1,\mu+1,\mu} = \frac{h^{2p+2}}{(p+1)(p!)^2 2^{2p}}\left\{(2p+1)\gamma_{\mu\mu0}\times \left[\left(\frac{\partial h}{\partial t}\right)^2 + \omega^2h^2\right] + \gamma_{\mu+1,\mu,0} + \frac{2p+1}{2p+2}\left(\frac{h^2\gamma_{\mu\mu0}}{2}\right)^{\{2\}}\right\} \quad (59)$$

These two equations determine $\gamma_{\mu+1,\mu,0}$ and $\beta_{2p+1,\mu+1,\mu}$ from β_{110} [itself given by Eq. (53)]. $\mathfrak{L}_{\mu+1,\mu}$ is now entirely determined. As sketched earlier, the next step would consist of completing $\mathfrak{L}_{20}$ obtained by integrating $\mathfrak{L}_{10}$ [Eq. (38)]. To a particular integral, we must add an arbitrary function

$$\gamma_{200}(h^2 - z^2)^{1/2} + \gamma_{202}z^2(h^2 - z^2)^{1/2} \quad \text{(a)}$$

to satisfy the matching principle, which will determine γ_{200} and γ_{202} (since there is now a term in $r^3\sin^3\theta$). Then we find the third diagonal ($\mathfrak{L}_{\mu+2,\mu}$; $\mu \geqslant 1$) by integrating $\mathfrak{L}_{\mu+1,\mu}$ and again adding an arbitrary function of the form given by expression (a). We would operate similarly for all the subsequent diagonals. Note that for the νth diagonal the arbitrary function [which for the second diagonal is given by expression (b)] is in general of the form,

$$\sum_{p=1}^{\nu+1}\gamma_{\nu,0,p-1}\epsilon_p z^{p-1}(h^2 - z^2)^{1/2} \qquad \epsilon_p = \frac{1-(-1)^p}{2} \quad \text{(b)}$$

this arbitrary function being necessary to satisfy the matching principle. In conclusion, we developed an unsteady wing theory that will be useful to compute the exact flowfield over thin wings. The results are given in closed form for the symmetric case, but even in the antisymmetric case there is no (theoretical) difficulty in carrying the computation to any order.

In a future paper, we shall investigate the singularities in planforms and deformations which can be studied with our theoretical model. Also, numerical applications will be presented to illustrate the advantages of our solution as well as its stability and convergence in practical cases.

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